Inherent Time-Frequency Coding in OFDM – – a Possibility for ISI Correction without a Cyclic Prefix?

Werner Henkel, Fangning Hu, and Ina Kodrasi

Jacobs University Bremen, Transmission Systems Group (TrSys)
E-mail {w.henkel, f.hu}@jacobs-university.de

***Abstract*— Zero-padded OFDM with unused carriers as guard band is readily observed as a time-frequency Reed-Solomon code with a Hamming distance in time and DFT domain. In principle, the time-frequency redundancy allows for correcting errors in time and DFT domain up the corresponding Hamming distances. Possible error patterns are impulsive and/or periodic. The RS code perspective appears to also provide a trade-off between guard interval length and DFT-domain redundancy up to the extreme of correcting inter-symbol interference (ISI) resulting from a completely missing guard interval. However, we have to be aware that the underlying problem is extremely ill-conditioned if we choose the redundancy in DFT-domain to be consecutive just as the ISI error pattern that we like to correct.**

I. Analog Time-Frequency RS Coding

An analog Reed-Solomon code is defined as follows:

Definition 1.1: A Reed-Solomon (RS) code of length N and minimum Hamming distance d_{Hm} is a set of vectors, whose components are the values of a polynomial $C(x) = x^l \cdot C'(x)$ of degree$\{C'(x)\} \leq K - 1 = N - d_{Hm}$ at positions z^k, with z being an element of order N from an arbitrary number field.

$$c = (c_0, \ldots, c_{N-1})\ , \quad c_i = C(x = z^i) \tag{1}$$

Writing out the polynomial in a sum formula and replacing z by $e^{\pm j2\pi/N}$ results in a standard DFT allowing us to observe OFDM with consecutively unused carriers as an RS code. Due to the symmetry of the DFT, however, consecutively zero-constrained time-domain samples are representing an RS code, as well, this time with the redundancy in time domain. With redundancy in time and DFT domain, we observe the following relation:

$$(c_0, \ldots, c_{K_T-1}, \underbrace{0, \ldots, 0}_{N-K_T}) =$$

$$= (C_0, \ldots, C_{K_F-1}, \underbrace{0, \ldots, 0}_{N-K_F}) \cdot \mathbf{W} \tag{2}$$

with $\mathbf{W}$ denoting the $N \times N$ DFT matrix. K_T and K_F are the numbers of non-zero elements in time and DFT domain, respectively. The sum of zeroed positions together should be less or equal to N, leading to the uncertainty principle of the DFT, namely

$$(N - K_T) + (N - K_F) \leq N \implies K_T + K_F \geq N \tag{3}$$

II. OFDM and Time-Frequency RS Coding

Instead of a cyclic prefix, one may think of zero padding as a guard interval, which we will later see as a part of a time-frequency RS code. Usually, such a zero-padded OFDM system would restore the cyclic property from the convolution components that will be present in the following guard interval. A missing (zero-padded) guard interval would not allow for such a correction, leading to artefacts at the beginning of the OFDM symbol. Having provided cyclically consecutive unused carriers as redundancy *would* allow for easy correction, since the inter-symbol interference would affect only the first components of the time-domain symbol, even allowing for erasure-decoding, since the actual error positions will be known. Even with a zero-padded interval as part of a time-frequency RS code, one could still use the DFT-domain redundancy to correct ISI and later make use of the time-domain redundancy to correct 'concentrated' errors in frequency domain, e.g., frequency-selective fading.

In order to illustrate the combined correction possibilities, we assume a number of erasures (errors at known positions) in the information parts, only, i.e., outside the syndromes of lengths $N - K_T = d_{HT} - 1$ and $N - K_F = d_{HF} - 1$ in time and DFT

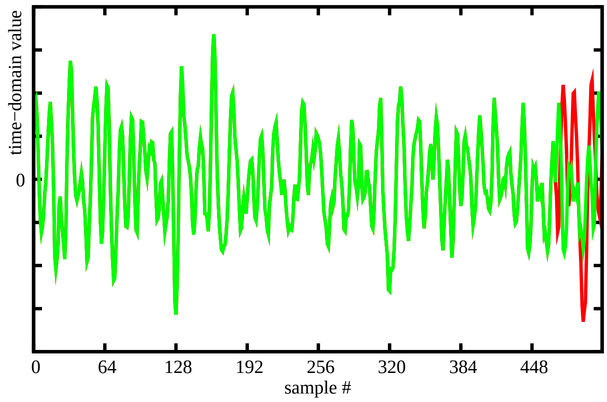


Fig. 1. ISI in case of ADSL on a 4 km 0.4 mm loop with NEXT disturbances from 8 line multiplexers and capacity-optimizing time-domain equalization

domain, respectively. We can then see the syndrome in time domain as only resulting from errors in the DFT-domain (information part) and in the same way, the syndrome in DFT domain as only resulting from errors in the time domain (info part). Thus, we should be able to correct the error pattern of Hamming weights below $d_{HT/F} - 1$ by separate least-squares solutions minimizing

$$(\mathbf{S}_{T/F} - \mathbf{E}_{F/T}\mathbf{W}'_{T/F})(\mathbf{S}_{T/F} - \mathbf{E}_{F/T}\mathbf{W}'_{T/F})^H \ , \quad (4)$$

leading to

$$\mathbf{E}_{F/T} = (\mathbf{S}_{T/F} \cdot \mathbf{W}'^{\,H}_{T/F}(\mathbf{W}'_{T/F}\mathbf{W}'^{\,H}_{T/F})^{-1} \ . \quad (5)$$

$\mathbf{E}_{F/T}$ denotes the error pattern (row vector) in DFT or time domain, respectively. With $\mathbf{W}'_{T/F}$, we mean a part of the DFT or IDFT matrix relating the error pattern and the syndrome in the other domain.

When errors are partly located in the corresponding syndromes, we may just omit these positions and the corresponding columns or rows of the DFT matrix $\mathbf{W}'_{T/F}$.

Although the procedure looks compelling, it is unfortunately extremely unstable. We simplify the task by choosing exactly as many redundant *cyclically consecutive* DFT positions (K_F syndromes) as we expect e erroneous signals at the beginning and/or end (*cyclically consecutive*) of the time-domain symbol ($e = K_F$). Figure 1 shows ISI in the case of ADSL when omitting the cyclic prefix after start-up, i.e., after having established time-domain equalization and bit allocation. One observes the effect at the end of the frame. This location is due to the chosen frame synchronization. One can, of course, also easily move the ISI to the beginning of the symbol.

Assuming the ISI to be at the end of the frame, in the case of OFDM, we would have a system of linear equations to be solved, which results from extracting a DFT submatrix. The submatrix is determined by the ISI positions and the syndrome locations as illustrated in Eq. 6.

$$\begin{aligned}(....|F_{K_F} \cdots F_{N-1}) &= \\ = (....|f_{N-e} \cdots f_{N-1}) &\cdot \quad (6)\end{aligned}$$

$$\left(\begin{array}{cccc|ccc}
1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\
1 & w & w^2 & w^3 & & & w^{N-1} \\
1 & w^2 & w^4 & & & & \\
1 & w^3 & & \ddots & & & \\
\vdots & & & & & & \\
\hline
1 & & & & & & \\
\vdots & & & & & & \\
1 & w^{N-1} & \cdots & & & & w^{(N-1)(N-1)}
\end{array}\right)$$

with $w = e^{-j2\pi/N}$. The right lower submatrix unfortunately is very ill-conditioned, so that an inversion is impossible. Figure 2 shows the condition numbers dependent on the matrix dimensions. Here we assumed the ISI to occur at the lower edge. The real matrix results for DMT are even worse for the ISI location on the opposite side.

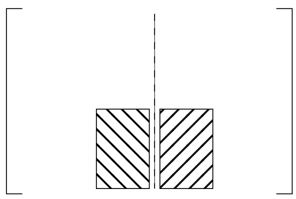
The condition number describes the propagation of relative errors when solving the set of linear equations, i.e.,

$$\frac{||\Delta f||}{||f||} \leq \underbrace{||\mathbf{W}'||||\mathbf{W}'^{-1}||}_{\kappa(\mathbf{W}')} \frac{||\Delta F||}{||F||} \ ,$$

where $\mathbf{W}'$ denotes the DFT submatrix. Note that $\kappa(\mathbf{W}') \geq 1$.

DMT, the baseband version of OFDM comes with a conjugacy constraint, meaning that we have two conjugate syndromes. We can sum them up together with the corresponding real parts of the columns of the DFT matrix to obtain a real-valued set of equations. The following may illustrate the situation

$$\begin{aligned}(...|F_{N/2-M} \cdots F_{N/2-1}|F_{N/2}|F_{N2+1} \cdots F_{N/2+M}|...) \\ = (....|f_{N-e} \cdots f_{N-1}) \cdot \quad (7)\end{aligned}$$



M denoting the number of consecutive syndromes.

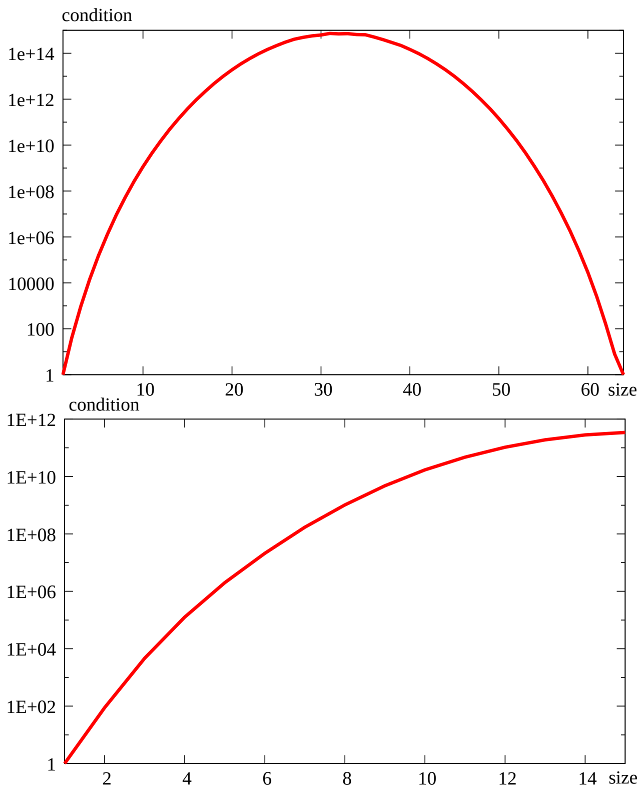


Fig. 2. Condition numbers of DFT submatrices for OFDM (upper plot) and real counterparts for DMT (lower plot) assuming the ISI to occur at the lower edge (DFT length: 64)

However, also this real counterpart has no better properties as far as the conditioning is concerned.

The use of consecutive high frequency bins would be ideal for ISI correction since in DSL, these carriers only have low SNRs and cannot carry a lot of information, anyhow. The average PSD in case of ADSL with and without a cyclic prefix reserving the upper 32 carriers as counterpart of the cyclic prefix of 32 is shown in Fig. 3. The effect of the ISI is clearly visible.

However, the extremely bad conditioning of using DFT submatrices with contiguous rows and columns does not allow for such a choice. A different selection of rows and/or columns (see, e.g., also [2] and references therein) will be required. Further work will be devoted to suitable selection schemes.

III. Conclusions

OFDM with a zero-padded guard interval as part of the OFDM symbol with consecutively unused carriers can be regarded as a time-frequency Reed-Solomon code, offering correction possibilities for combined inter-symbol/impulse-noise [1] and narrow-band interferences.

Unfortunately, consecutively used redundancy and consecutive error pattern in the other domain leads

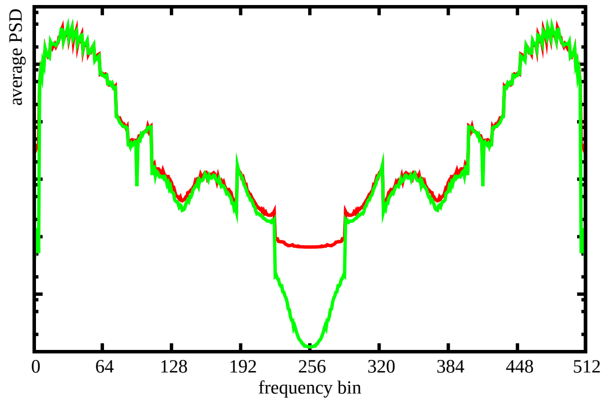


Fig. 3. Power spectral density with (green) and without (red) cyclic prefix
(ADSL on a 4 km 0.4 mm loop with NEXT disturbances from 8 line multiplexers and capacity-optimizing time-domain equalization)

to an ill-conditioned problem which does not allow to determine the errors even if the error positions are known.

Further work will concentrate on improving the stability, i.e., lowering the condition number of the underlying system of equations. Utilizing frequency-domain redundancy for ISI correction still appears to be an interesting alternative to a cyclic prefix, although at the expense of a higher complexity.

References

[1] W. Henkel and F. Hu, “OFDM and Analog RS/BCH Codes,” *OFDM-Workshop 2005*, Hamburg, Aug. 31 - Sept. 1, 2005.

[2] P.J.S.G. Ferreira, J.M.N. Vieira, “Stable DFT Codes and Frames,” *IEEE Signal Processing Letters*, Vol. 10, No. 2, pp. 50-53, Feb. 2003.